

# A non-perturbative renormalization group analysis of stochastic spiking networks

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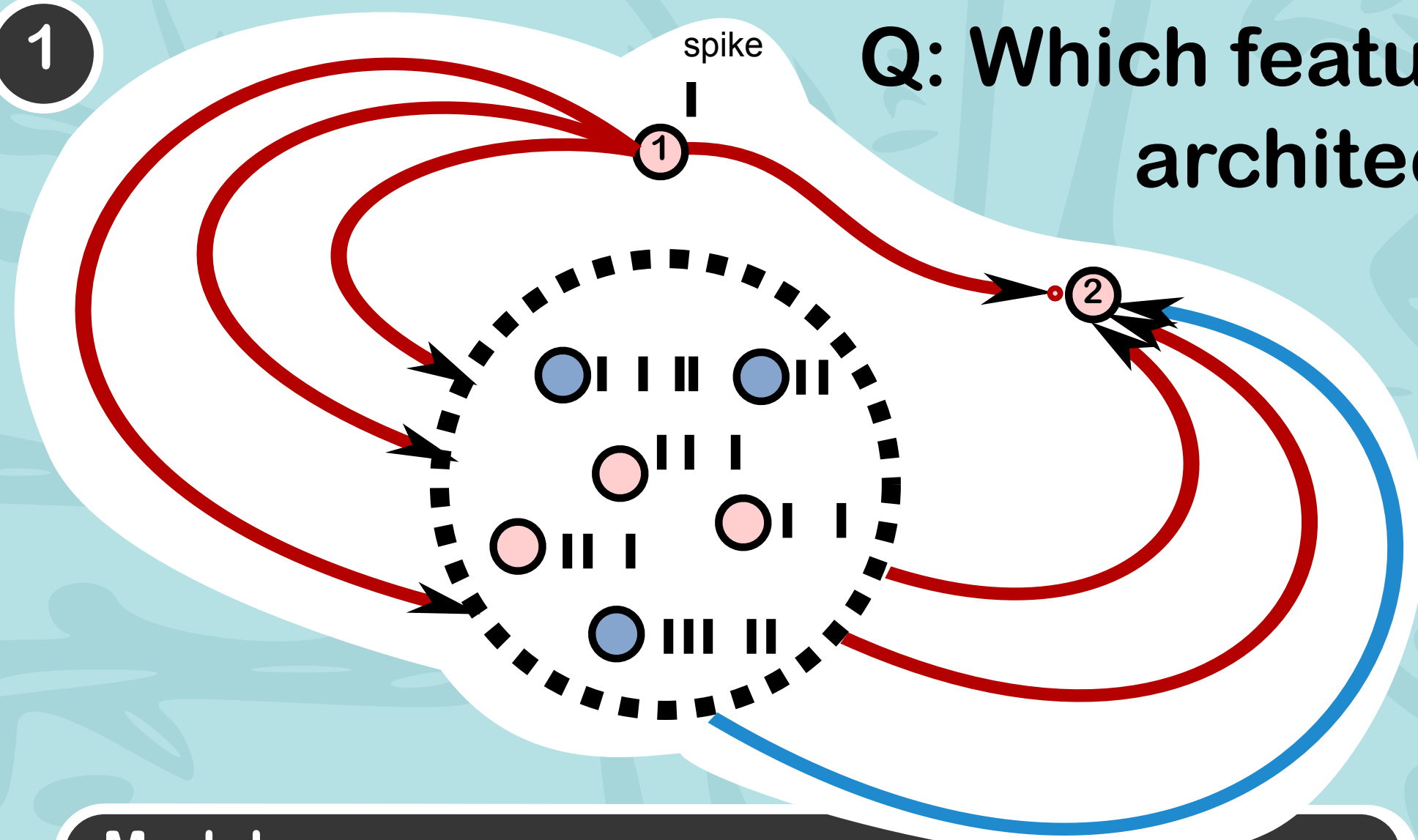
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Understanding which features of network architecture are most important for shaping the collective activity patterns of neural circuits is a major goal of theoretical neuroscience. For example, what network properties influence critical exponents in neural circuits at critical points? Answering this question with theory requires tools from the renormalization group, which to date have not been applied to spiking network model used in neuroscience due to the all-or-nothing nature of spiking. Here we apply to RG to a stochastic spiking model and show that spontaneously active networks are controlled by a critical point in the Ising universality class..

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Q: Which features of network architecture are most important for shaping activity patterns?



Model

- Soft-threshold leaky-integrate & fire dynamics

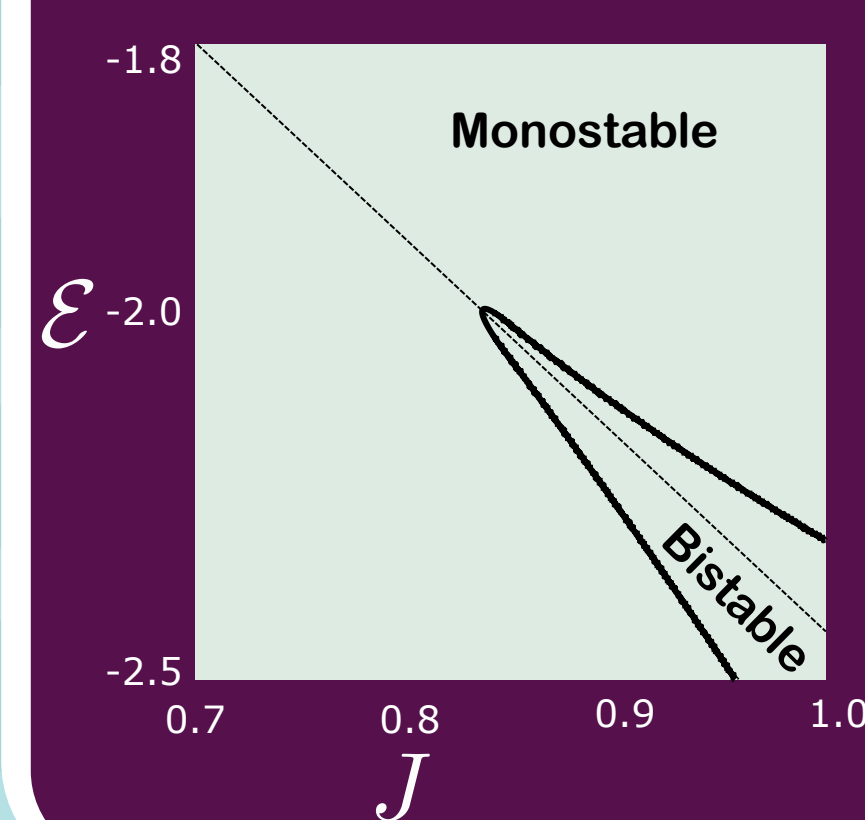
$$\tau \frac{dV_i(t)}{dt} = -V_i(t) + \mathcal{E}_i + \sum_j J_{ij} \hat{n}_j(t)$$

- Stochastic spiking

$$\hat{n}_i(t) dt \sim \text{Pois}[\phi(V_i(t)) dt]$$

- membrane potential  $V_i(t)$
- spike train  $\hat{n}_i(t)$
- synaptic coupling  $J_{ij}$
- firing nonlin.  $\phi(V_i(t))$
- input bias  $\mathcal{E}_i$
- time constant  $\tau$

Mean-field theory of network dynamics



- Mean-field theory predicts transition from monostable to bistable firing rates
- Critical line  $\mathcal{E}_c(J)$  along which  $\langle V \rangle = \theta$ , separates high & low firing rate regions
- Assumptions:
  - Synaptic weights with leading homogeneous mode
  - Bounded firing nonlinearity

Beyond mean-field theory: Non-perturbative Renormalization Group (NPRG)

- Technique from statistical physics for analyzing collective behavior of interacting systems
- Key object: "Average Effective Action" (AEA) contains all info about statistics & response of a stochastic process
- Mean-field approximation of the AEA (using auxiliary spike & membrane fields  $\tilde{n}, \tilde{V}$ ):

$$\Gamma_{\text{MFT}} = \sum_{i=1}^N \int_{-\infty}^{\infty} dt \left\{ \tilde{V}_i(t) \left[ \tau \frac{dV_i(t)}{dt} + V_i(t) - \mathcal{E}_i - \sum_j J_{ij} \hat{n}_j(t) \right] + \tilde{n}_i(t) \hat{n}_i(t) - \left( e^{\tilde{n}_i(t)} - 1 \right) \phi(V_i(t)) \right\}$$

- Key idea: Create family of AEAs  $\Gamma_\Lambda$ , parametrized by  $\Lambda$ , to interpolate from mean-field theory to the true AEA
- The AEAs obey an exact differential equation amenable to approximations by projecting solution into solvable subspace

$$\partial_\Lambda \Gamma_\Lambda = \frac{1}{2} \text{Tr} \left[ \partial_\Lambda R_\Lambda \cdot \left[ \Gamma_\Lambda^{(2)} + R_\Lambda \right]^{-1} \right]$$

where Tr denotes a super-trace over fields, neurons, & time,  $R_\Lambda$  regulates the interpolation, &  $\Gamma_\Lambda^{(2)}$  is a matrix of 2nd derivatives with respect to the fields

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Key results

- Successful adaptation & application of NPRG methods to stochastic spiking networks
- Fixed point analysis of RG flow suggests that spontaneous networks with homogeneous modes are in the Ising universality class
- The  $Z_2$  Ising symmetry is an emergent symmetry in spiking networks, does not hold in general

Next steps

- Spinodal critical points & metastability in finite-sized networks: Lee-Yang universality class
- Rectified nonlinearities & avalanches: Directed percolation universality class
- Effects of "disorder": Griffiths phases?

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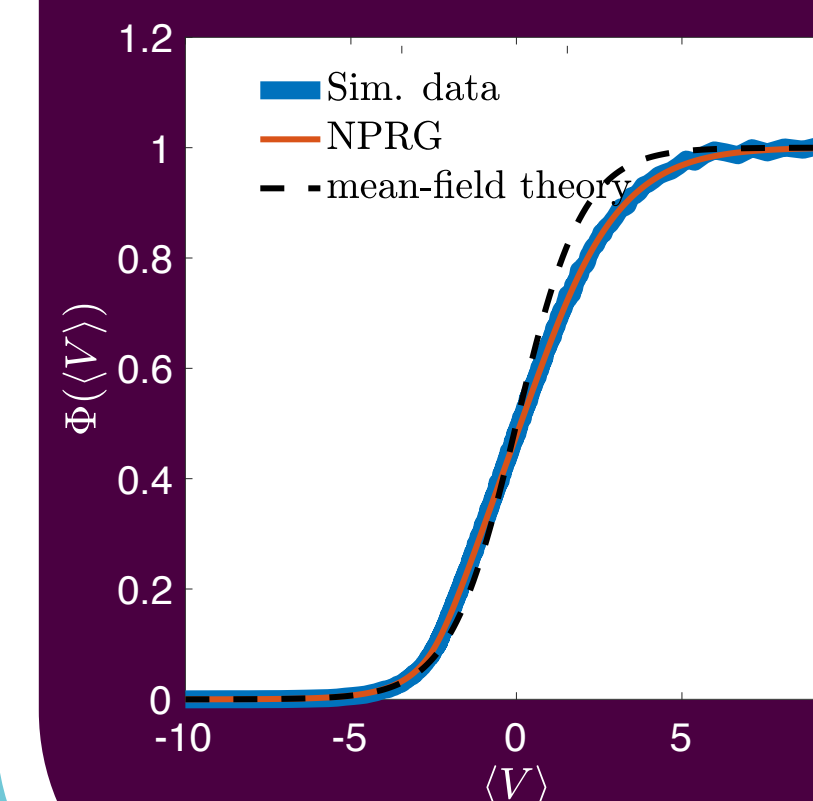
Critical dynamics predicted by an effective nonlinearity

- RG analysis: only spiking statistics are renormalized
- Dimension  $d$  of network is determined by scaling of eigenvalue distribution of  $J_{ij}$
- Mean membrane potential follows mean-field dynamics with effective nonlinearity  $\Phi(V)$

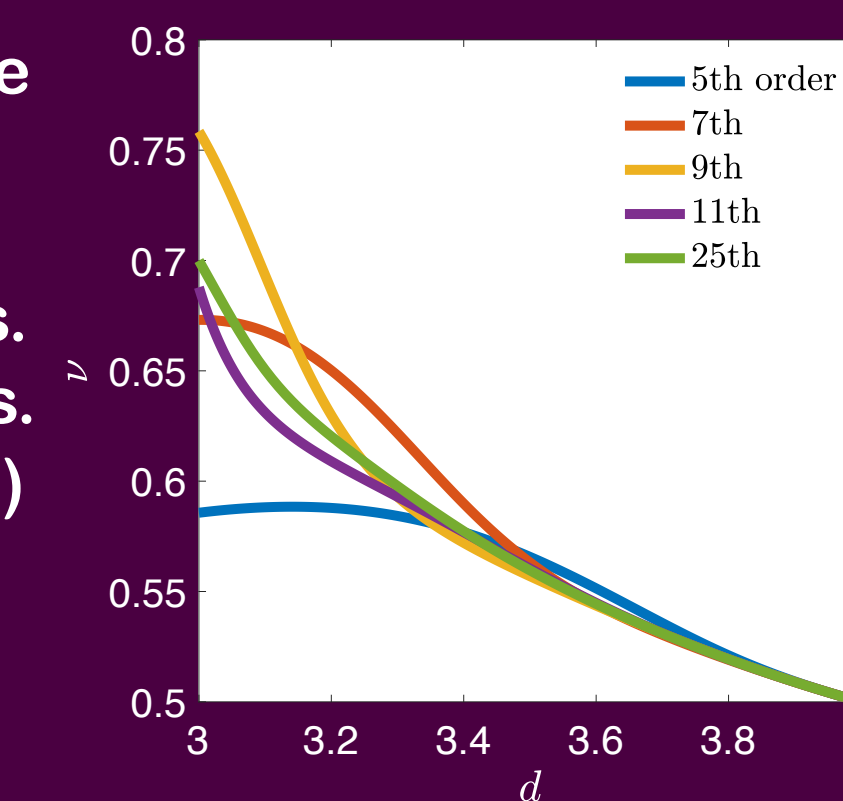
$$\Phi(\langle V \rangle) = \Phi(\theta) + ((2d-1)J)^{-1} (\langle V \rangle - \theta) + |J_c - J|^{\nu(d+2)/2} f\left(\frac{\langle V \rangle - \theta}{|J_c - J|^{\nu(d+2)/2}}\right)$$

where  $\theta$  is the running set point,  $d$  is the spectral dimension,  $\nu$  is a critical exponent, and  $f$  is a scaling function

- Can estimate  $\Phi(V)$  numerically by solving RG equations &  $\nu$  by finding fixed point of flow



- $d = 3$  excitatory lattice
- Parametric plot of population & trial-averaged spike-rate vs. membrane potential vs. time, starting from  $V(0) = \pm 10$



- Sequence of approximations of  $\nu$  vs.  $d$  appear to be converging close to Ising model exponent

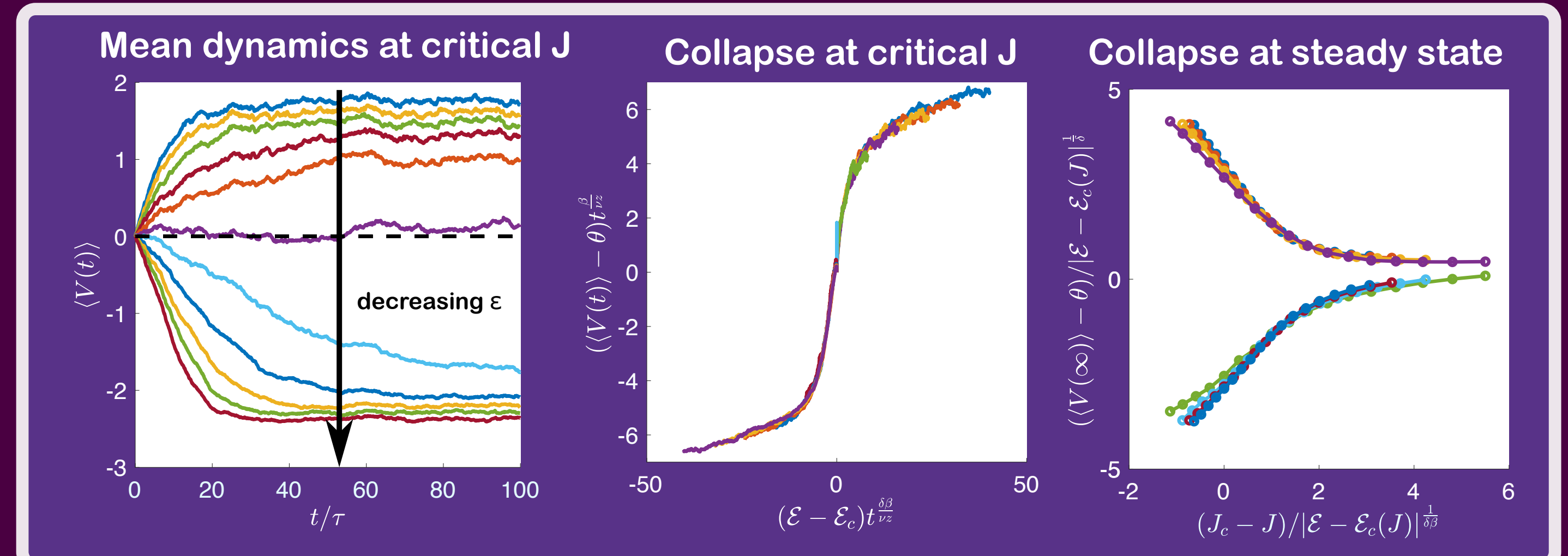
- RG fixed point has an emergent  $Z_2$  symmetry not present in microscopic model
- Results suggest model is in Ising model universality class

Simulations of critical network dynamics

- Check universality class with simulations,  $N = 14^3$ ,  $T_{\text{max}} = 100\tau$ ,  $d = 3$  excitatory lattice
- Scaling theory of membrane potential dynamics near critical point:

$$\langle V(t) \rangle - \theta \sim t^{-\frac{\beta}{\nu z}} F\left(\frac{\mathcal{E} - \mathcal{E}_c(J)}{t^{\frac{\delta\beta}{\nu z}}}, t |J_c - J|^{\nu z}\right)$$

where  $F$  is determined implicitly by solving the dynamics of  $\langle V(t) \rangle$



- using Ising model critical exponent estimates
- $\beta = 0.33$ ,  $\delta = 4.79$ ,  $\nu = 0.63$ ,  $z = 2.02$

- Scaling collapse is consistent with Ising universality class!

References:

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